

# Notes on dual-critical graphs

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**Abstract:** We define dual-critical graphs as graphs having an acyclic orientation, where the indegrees are odd except for the unique source. We have very limited knowledge about the complexity of dual-criticality testing. By the definition the problem is in NP, and a result of Balázs and Christian Szegedy [11] provides a randomized polynomial algorithm, which relies on formal matrix rank computing. It is unknown whether dual-criticality test can be done in deterministic polynomial time. Moreover, the question of being in co-NP is also open.

We give equivalent descriptions for dual-critical graphs in the general case, and further equivalent descriptions in the special cases of planar graphs and 3-regular graphs. These descriptions provide polynomial algorithms for these special classes. We also give an FPT algorithm for a relaxed version of dual-criticality called  $k$ -dual-criticality.

**Keywords:** dual-critical, factor-critical, ear decomposition, FPT algorithm

## 1 Introduction

The name and definition of dual-critical graphs was introduced by András Frank [1] based on the paper of Szegedy and Szegedy [11]. He showed that this class is very interesting due to the following reason. Let  $G(V, E)$  be a simple graph. We denote by  $e_G(X)$  the number of incident edges for any set  $X \subseteq V$ . The function  $e_G$  defines a polymatroid  $\mathcal{P}_G$  on  $V$ . A graph is *dual-critical*, if and only if  $\mathcal{P}_G$  has a vertex  $x$  such that all coordinates of  $x$  are odd.

An orientation  $\vec{G} = (V, \vec{E})$  of a graph  $G$  is called *acyclic* if it does not have directed cycles. A graph  $G$  is *dual-critical* if it has an acyclic orientation such that all vertices except one have an odd indegree.

This definition might look factitious, but as we will see later, when looking for graphs with certain parity constrained acyclic orientations, it is always possible to reduce the problem to the dual-criticality of a slightly altered graph.

In every acyclic orientation there is a source vertex  $v$ . It has an even indegree (0). Thus an orientation of a dual critical graph that satisfies the above conditions has exactly one vertex (the source) that has an even indegree, and  $|V(G)| - 1$  vertices that have odd indegrees. Consequently a dual-critical graph is always loopless and connected.

It is a known fact that acyclic graphs have a *topological ordering*. In a topological ordering the source vertex comes first. The orientation of the edges is determined by the order of their endpoints: the source of an arc always precedes its target. Let us take a topological ordering of an orientation described above. Beginning with the second vertex, every vertex has an odd number of predecessors to which it is connected. Consequently, the class of dual-critical graphs can be characterized as the graphs that can be built by taking a single vertex, and adding new vertices connected to the previous ones by an odd number of edges. This description shows that the problem is in NP. Such an ordering will be called a *good ordering*, and the orientation defined by a good ordering will be called a *good orientation*.

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A directed graph is *rooted connected* if it has a vertex  $r$  called root, from which there is a path to any other vertex.

**Remark 1** *It is easy to see that a good orientation of a dual-critical graph is rooted connected, with the source vertex as root. Indeed, except the first vertex (the source or the root) in the good ordering, every vertex has at least one incoming arc. So one could construct a backward path from any vertex to the root by going backwards on incoming arcs.*

Organization of the paper: The second section introduces the basic properties of dual-critical graphs. We also present a related class called super-dual-critical graphs. In Section 3 we deal with the background of the terminology, which lies in planar dual-critical graphs. Section 4 describes the randomized algorithm of Balázs Szegedy and Christian Szegedy. The next section deals with 3-regular graphs. The main theorem of this section shows that dual-critical graphs coincide with many graph classes when restricted to 3-regular graphs. One of these classes is upper-embeddable graphs, providing us a deterministic polynomial algorithm for the 3-regular case. In the final section we define  $k$ -dual-criticality. András Frank asked the original question whether an FPT algorithm for testing  $k$ -dual-criticality can be given. We present the first FPT algorithm for this problem.

With the exception of Section 7, many proofs are omitted, but all of these proofs can be found in [5].

## 2 Basic properties of dual-critical graphs

**Notation:** For a subset  $X$  of vertices,  $i(X)$  denotes the number of edges induced by  $X$ . The symmetric difference of  $X$  and  $Y$  is denoted by  $X \oplus Y$ . When writing congruences, the notation of the modulus will be omitted if it is 2, e.g.,  $a \equiv b$  means that  $a$  and  $b$  have the same parity.

If  $|V| \not\equiv |E|$  holds for a graph  $G(V, E)$ , we say that  $G$  has *good parity*, otherwise, if  $|V| \equiv |E|$ , we say that  $G$  has *bad parity*.

**Definition 2 (T-odd)** *Let  $T \subseteq V$ . An orientation of a graph  $G = (V, E)$  is  $T$ -odd if all vertices in  $T$  have odd indegree, and all vertices in  $V - T$  have even indegree.*

**Theorem 3** *The following statements are equivalent for any graph  $G = (V, E)$ :*

- (1)  $G$  is dual-critical.
- (2) For any given  $v \in V(G)$  there is an acyclic orientation, in which all the indegrees are odd except for  $v$ .
- (3) The graph has good parity and for every set  $T \subsetneq V$  with  $|T| \equiv |E|$ , there exists a  $T$ -odd acyclic orientation.
- (4) Either  $G$  is the graph consisting of one vertex, or it has a two-class partition  $V = A \cup B$ , such that  $G[A]$  and  $G[B]$  are dual-critical, and the cut  $E(A, B)$  defined by  $A$  and  $B$  has an odd number of edges (i.e.,  $d(A, B) = |E(A, B)|$  is odd).

PROOF:

(1) $\Rightarrow$ (4) Take a good ordering of  $G$ . Let  $w$  be the last vertex in the ordering. Choose  $A = V(G) - w$ ,  $B = \{w\}$ .

(4) $\Rightarrow$ (3) We use induction on the number of vertices. If  $|T \cap A| \not\equiv i(A)$  and  $|T \cap B| \not\equiv i(B)$ , then  $|T| \not\equiv i(A) + i(B) + d(A, B) = |E|$ , a contradiction. Wlog. one can suppose  $i(A) \equiv |T \cap A|$ . By induction, as  $G[A]$  is dual-critical, we can take an acyclic orientation of  $G[A]$  in which the vertices of  $T \cap A$  have an odd indegree, and the vertices of  $A - T$  have an even indegree. Direct the edges of  $E(A, B)$  towards  $B$ .

As  $|T \cap A| \equiv i(A)$  we have that  $|T \cap B| \not\equiv i(B)$ . Let  $Z$  be the set of vertices in  $B$  that have an odd number of incoming edges from  $A$ , and let  $T' = (T \cap B) \oplus Z$ . As  $|Z|$  is odd,  $|T'| \equiv i(B)$ .

Now one can use induction for  $G[B]$  and  $T'$ , and fix an acyclic  $T'$ -odd orientation of  $G[B]$ . It is easy to check that the resulting orientation of  $G$  is also acyclic and moreover it is  $T$ -odd.

(3) $\Rightarrow$ (2) We may use (3) for  $T = V(G) - v$ .

(2) $\Rightarrow$ (1) Obvious.  $\square$

**Proposition 4** *The following operations do not change dual-criticality, i.e., a graph is dual-critical, if and only if using any of these operations results in a dual-critical graph:*

- (1) Deletion of two parallel edges,
- (2) Insertion of two parallel edges between two arbitrary vertices,
- (3) Division of an edge by adding a vertex in the middle,
- (4) Contraction of an edge that has an endvertex with degree 2.

Using the operations (1) and (3) from Proposition 4 one can make any graph simple by dividing loops by two vertices (so a triangle is made) and eliminating parallel edge pairs. We might cut a connected graph this way, but in that case Proposition 4 states that the original graph was not dual-critical.

**Definition 5 (Super-dual-critical)** *A graph is called super-dual-critical, if for any vertex  $v \in V(G)$ , the graph  $G - v$  is dual-critical.*

**Proposition 6** *In super-dual-critical graph either all vertex degrees are odd, or all vertex degrees are even. The degree parity is the same as the parity of  $|E(G)| - |V(G)|$ .*

PROOF: For an arbitrary vertex  $v$  the graph  $G - v$  has good parity, thus

$$|V(G)| - 1 \not\equiv |E(G)| - d(v) \Rightarrow d(v) \equiv |E(G)| - |V(G)|. \quad (1)$$

$\square$

**Corollary 7** *A super-dual-critical graph is dual-critical if and only if it has good parity, or equivalently, a super-dual-critical graph is dual-critical if and only if every vertex has odd degree.*

PROOF: Let  $G$  be a super-dual-critical graph. If it has bad parity, then by Proposition 6 all degrees are even, so it cannot be dual-critical. If  $G$  has good parity, then all degrees are odd. Delete an arbitrary vertex  $v$ . The graph  $G - v$  is dual-critical, hence  $G$  is dual-critical as well, since it can be obtained from  $G - v$  by adding a vertex that has odd degree.  $\square$

**Proposition 8** *The graph  $G$  has a  $T$ -odd acyclic orientation for every  $T \subsetneq V(G)$  for which  $|T| \equiv |E(G)|$  if and only if  $G$  is dual-critical or  $G$  is super-dual-critical.*

Finally, the following is a proposition which shows that a question about acyclic orientations with parity constraints can be viewed as dual-criticality of a slightly changed graph.

**Proposition 9 (Beáta Faller)** *A graph  $G = (V, E)$  has a  $T$ -odd acyclic orientation for an arbitrary  $T \subseteq V$ , if and only if the graph  $G'$  obtained by adding a vertex  $v$  and connecting it to all vertices in  $V - T$  is dual-critical.*

PROOF: If  $G$  has such an orientation, then directing the edges away from  $v$  will give a good orientation for  $G'$ . If  $G'$  is dual-critical, then it has a good orientation in which the source vertex is  $v$ . This orientation is  $T$ -odd and acyclic in  $G$ .  $\square$

### 3 Planar case, motivations

Before talking about dual-critical graphs in more detail, some background information should be provided about the term 'dual-critical'. In this section we are going to use matroids. Ample introductory and advanced material can be found on them. See e.g., [9]. In this section we also omit some basic definitions, see [5], and graphs may have parallel edges and loops.

There is a well-known result about factor-critical graphs and ear decompositions.

**Theorem 10** [7] *A graph is factor-critical if and only if it has an odd ear decomposition, i.e., an ear decomposition in which all ears have an odd number of edges.*

We will use the following variant of this theorem.

**Theorem 11** *A graph is factor-critical if and only if it can be built from a single vertex using the following operations:*

- Addition of an edge between two vertices or addition of a loop,
- Division of an edge into path of length three using two new vertices.

**Theorem 12** *The planar dual of a planar factor-critical graph  $G$  is always dual-critical. (So if there are multiple dual graphs depending on the planar embedding of  $G$ , then all of them are dual-critical.)*

**Proposition 13** *Let  $G$  be a factor-critical graph. The graph  $G'$  that is obtained from  $G$  by blowing an odd cycle into a vertex  $v$  is factor-critical.*

**Theorem 14** *The dual of a planar dual-critical graph  $G$  is always factor-critical. (If there are multiple dual graphs depending on the planar embedding of  $G$ , then all of them are factor-critical.)*

Since dualization and factor-criticality test can be done in polynomial time, we arrive at the following corollary.

**Corollary 15** *There is a polynomial time algorithm for deciding dual-criticality if the graph is planar.*  
□

Theorems 12 and 14 show that dual-criticality and factor-criticality are dual concepts. Our next goal is to show that Theorems 12 and 14 can be generalized.

**Remark 16** *It is known that a graph is 2-connected if and only if it has an open ear-decomposition, i.e., an ear decomposition where we begin with a cycle, and the two ends of an ear cannot coincide. This statement can be ported for factor-critical graphs. A 2-connected graph is factor-critical if and only if it can be obtained from an odd cycle by adding odd length open ears.*

Now we state a well-known result from matroid theory. A proof can be found in section 2.3 in [9].

**Proposition 17** *The graphic matroid of the dual of a planar graph is isomorphic to the cographic matroid of the graph. (Equivalently: the dual graph's graphic matroid is the dual of the graph's graphic matroid.)*

**Definition 18** [11] *A sequence of circuits  $\{C_0, C_1, \dots, C_k\}$  of the matroid  $M = (S, \mathcal{F})$  is called an ear-decomposition if*

- (1)  $C_i - (\bigcup_{j=0}^{i-1} C_j)$  is not empty for all  $1 \leq i \leq k$
- (2)  $C_i \cap (\bigcup_{j=0}^{i-1} C_j)$  is not empty for all  $1 \leq i \leq k$
- (3)  $C_i - (\bigcup_{j=0}^{i-1} C_j)$  is a circuit in  $M/(\bigcup_{j=0}^{i-1} C_j)$  for all  $1 \leq i \leq k$
- (4)  $\bigcup_{i=0}^k C_i = S$

*An ear is a set  $C_i - (\bigcup_{j=0}^{i-1} C_j)$ .*

This definition is the matroid equivalent of the open-ended ear-decomposition which is described in Remark 16. It follows that a 2-connected graph is factor-critical if and only if its graphic matroid has an odd ear-decomposition.

We need two basic lemmas from matroid theory. The notation  $M^*$  indicates the dual matroid of  $M$ , and  $M/Z$  is used for the contraction of the subset  $Z$ .

**Lemma 19** (Theorem 8.3 in [6]) *Let  $M$  be a matroid on the set  $S$ . Then for any  $Z \subseteq S$  the following holds:*

$$(M/Z)^* = M^* - Z \text{ and } M^*/Z = (M - Z)^* \quad (2)$$

**Lemma 20 (Proposition 2.3.1 in [9])** *Let  $M = (E(G), \mathcal{F})$  be the graphic matroid of  $G$ . The set  $Z \subseteq E(G)$  is a cycle in  $M^*$  if and only if it is a proper cut in  $G$ .*

**Proposition 21** *A 2-connected graph is dual-critical if and only if its cographic matroid has an odd ear decomposition.*

## 4 A randomized algorithm by Balázs and Christian Szegedy

**Definition 22** ([11]) *Let  $M$  be a connected, bridgeless matroid. We denote by  $\varphi(M)$  the minimal possible value of the number of even ears in an ear-decomposition of  $M$ . If  $M$  is bridgeless but not connected, we define  $\varphi(M)$  to be the sum of  $\varphi(K)$  over all blocks  $K$  of  $M$ . In particular  $\varphi(M) = 0$  if and only if every block of  $M$  has an odd ear-decomposition.*

**Theorem 23 (Szegedy–Szegedy, Theorem 10.8 in [11])** *Let  $M$  be a matroid that is representable over a field of characteristic 2. There is a randomized polynomial algorithm which computes  $\varphi(M)$ .*

This gives a randomized polynomial algorithm for deciding dual-criticality. We can represent the cographic matroid of graphs over a field of characteristic two: a graph is dual-critical if and only if  $\varphi(M^*(G)) = 0$ .

We would like to outline this algorithm for cographic matroids. Let  $T$  be the edge set of a spanning tree of our graph. We associate independent indeterminates with each edge of  $T$ :  $x_e$  for all  $e \in T$ . The tree edges of the fundamental cycle of  $i \in E(G) - T$  will be denoted by  $T_i$ . Let  $A = (a_{i,j})$  be the following matrix ( $i \in E(G) - T$  and  $j \in E(G) - T$ ):

$$a_{ij} = \sum_{e \in T_i \cap T_j} x_e.$$

If the fundamental cycles have no common edge (the sum is empty), then the matrix entry is 0. The corank of  $A$  is equal to  $\varphi(M^*)$  by the theorem of Szegedy–Szegedy ([11]). So  $G$  is dual-critical if and only if  $\det(A)$  is not the constant zero polynomial. (The determinant is defined over the representing field.)

After choosing a large enough field of characteristic 2, this can be decided using the Schwarz–Zippel lemma ([10]), which provides a polynomial randomized algorithm.

## 5 3-regular dual-critical graphs

### 5.1 Equivalent descriptions

A 3-regular graph on  $n$  vertices have  $\frac{3n}{2}$  edges, so  $n$  is even. If the graph is also dual-critical, then it must have a good parity, thus  $n = 4k + 2$  for some integer  $k$ .

**Theorem 24 \*** *The following are equivalent for any 3-regular graph  $G = (V, E)$  which has  $4k + 2$  vertices.*

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\*Beáta Faller and Ervin Győri had the original idea that dual-criticality is equivalent to description (2) in the 3-regular case.

- (1)  $G$  is dual-critical.
- (2) There are  $k + 1$  independent vertices, such that their deletion leaves a connected graph.
- (3) There are  $k + 1$  vertices whose deletion leaves a forest.
- (4) There are some independent vertices whose deletion leaves a tree.
- (5) There is a spanning tree, for which the deletion of the tree's edges makes a graph in which every component has an even number of edges.
- (6) There is an  $r$ -rooted connected orientation for every vertex  $r \in V$ , in which all vertices but  $r$  have an odd indegree.
- (7) For every partition  $\mathcal{P}$  of  $V$ 

$$e(\mathcal{P}) \geq |\mathcal{P}| + \text{bp}(\mathcal{P}) - 1 \quad (3)$$
holds, where  $e(\mathcal{P})$  denotes the number of edges between different classes of  $\mathcal{P}$  and  $\text{bp}(\mathcal{P})$  denotes the number of classes in  $\mathcal{P}$  spanning a subgraph which has bad parity.
- (8)  $G$  is upper-embeddable. <sup>†</sup>

**Theorem 25 (Furst, Gross, McGeoch [4])** *There is a polynomial algorithm that decides whether a graph is upper-embeddable or not. It runs in  $O(\text{end} \log^6 n)$  time where  $e$ ,  $n$  and  $d$  denote the number of edges, the number of vertices and the maximum degree respectively.*

**Corollary 26** *There is an algorithm for deciding dual-criticality in the 3-regular case which runs in  $O(n^2 \log^6 n)$  time.*

## 6 A relaxed version of dual-criticality – $k$ -dual-critical graphs

**Definition 27 ( $k$ -dual-critical, [1])** *A graph  $G(V, E)$  is called  $k$ -dual-critical if  $V$  has a partition  $\mathcal{P}$  into  $k$  non-empty subsets such that the contraction of all partition classes results in a dual-critical graph. (In this contraction the loops are deleted, but the parallel edges are preserved.)*

Equivalently, a graph is  $k$ -dual critical, if and only if its vertex set has a partition  $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$  such that  $d(\cup_{j=1}^{i-1} P_j, P_i)$  is odd for each  $2 \leq i \leq k$ , (recall that  $d(A, B)$  denotes the number of edges between two disjoint vertex sets  $A, B$ ). We say that a vertex partition is a good  $k$ -partition if it satisfies the condition in the above definition. We use the term good ordering for partition classes of a good  $k$ -partition as we did for vertices of a dual-critical graph.

From the definition it is trivial that a graph is dual-critical if and only if it is  $|V|$ -dual-critical. It is also easy to observe that every graph is 1-dual-critical. Moreover, if  $G$  is  $k$ -dual-critical, then it is  $\ell$ -dual-critical for each  $1 \leq \ell \leq k$ , since one can take a good ordering of the partition classes of a good  $k$ -partition, and unify the first  $k - \ell$  classes to get a good  $\ell$ -partition.

We denote by  $\text{maxdc}(G)$  the biggest number  $k$  for which  $G$  is  $k$ -dual-critical. It is easy to verify that  $G$  is Eulerian if and only if  $\text{maxdc}(G) = 1$ . *Optimal* partition of a graph will refer to a good  $\text{maxdc}(G)$ -partition. The following proposition is easy to verify:

**Proposition 28** *The minimal number of vertex pairs that need to be contracted in  $G$  to get a dual-critical graph is  $|V(G)| - \text{maxdc}(G)$ .*

**Proposition 29** *Let  $G$  be any simple graph. If the graph spanned by a partition class in a good  $\ell$ -partition is non-Eulerian, then this class can be divided into two subclasses to get a good  $(\ell+1)$ -partition. Consequently in an optimal partition of  $G$  every class spans an Eulerian subgraph.*

<sup>†</sup>For the definition of upper-embeddability see [5]

PROOF: Suppose that a class  $P_i$  of a good partition  $P_1, P_2, \dots, P_\ell$  spans a non-Eulerian subgraph. This graph has an odd proper cut with sides  $P'_i$  and  $P''_i$ . Since  $d(P_i, \cup_{j=1}^{i-1} P_j)$  is odd, one of  $d(P'_i, \cup_{j=1}^{i-1} P_j)$  and  $d(P''_i, \cup_{j=1}^{i-1} P_j)$  is odd, and the other one is even. Suppose  $d(P'_i, \cup_{j=1}^{i-1} P_j)$  is odd. Then the partition  $(P_1, P_2, \dots, P_{i-1}, P'_i, P''_i, P_{i+1}, \dots, P_\ell)$  is a good  $(\ell+1)$ -partition of  $G$ .  $\square$

**Theorem 30** *Every graph has an optimal partition where all classes contain 1 or 2 vertices except possibly the first class.*

PROOF: Let  $P_1, P_2, \dots, P_k$  be a good ordering of an optimal partition  $\mathcal{P}$ , and let  $Q_i = \cup_{j=1}^i P_j$ .

Starting from  $P_k$  we will show that if  $|P_i| > 2$  then some vertices can be moved from  $P_i$  to  $P_{i-1}$  (for each  $i = k, k-1, \dots, 2$ ) leaving at most 2 vertices in  $P_i$ . Note that this operation preserves the parity of  $d(Q_{j-1}, P_j)$  if  $j \notin \{i-1, i\}$ . We fix an  $i \geq 3$  and observe that a set of vertices  $S \subsetneq P_i$  can be moved to  $P_{i-1}$  if and only if

$$d(S, P_i - S) \equiv d(S, P_{i-1}) \quad (4)$$

$$d(S, Q_{i-2}) \equiv 0. \quad (5)$$

Since  $\mathcal{P}$  is optimal, by Proposition 29 the graph spanned by  $P_i$  is Eulerian, thus all its cuts are even; so our conditions for a set  $S$  to be movable are  $d(S, P_{i-1}) \equiv d(S, Q_{i-2}) \equiv 0$ .

It is sufficient to show that if  $|P_i| \geq 3$  then there is a non-empty movable  $S \subsetneq P_i$ . For a vertex set  $R \subsetneq P_i$  let  $z_R$  be a 2-dimensional vector over the two element field representing the parity of  $d(Q_{i-2}, R)$  and  $d(P_{i-1}, R)$ . Note that  $S$  is movable if and only if  $z_S = (0, 0)$ .

Let  $A, B, C$  be any 3-partition of  $P_i$  where all the classes are non-empty. (Such a partition exists because  $|P_i| \geq 3$ .) If one of  $z_A, z_B$  or  $z_C$  is  $(0, 0)$ , then the corresponding set is movable. Suppose there are two equal vectors among  $z_A, z_B$  and  $z_C$ , eg.  $z_A = z_B$ . Then  $z_A + z_B = (0, 0)$ , thus  $S = A \cup B$  is movable. The only remaining case is that the three vectors are  $(0, 1), (1, 1)$  and  $(1, 0)$  in some order. But this means that  $z_A + z_B + z_C = (0, 0)$ , thus  $d(P_i, Q_{i-1}) \equiv 0$ , a contradiction.

If  $i = 2$ , then let  $v$  be any vertex in  $P_2$  for which  $d(P_1, v)$  is odd. (Such vertex exists because  $d(P_1, P_2)$  is odd.) The set  $P_2 - v$  is moveable, since  $d(P_2 - v, P_1) \equiv 0$ .  $\square$

We call a good  $k$ -partition *left-aligned* if all its classes except possibly the first one have either 1 or 2 vertices. For a constant  $k$ , the  $k$ -dual-criticality of a graph can be determined in polynomial time. Consider the following algorithm that decides  $k$ -dual-criticality:

**Recursive  $k$ -dual-critical** algorithm:

If  $k \leq 1$ , then we return true. For each odd degree vertex  $v$  if **Recursive  $k$ -dual-critical**( $k-1, G-v$ ) is true, then we return true. Then we take each vertex pair  $v, w$  such that  $d(v) + d(w) \equiv 1$ , and we return true if **Recursive  $k$ -dual-critical**( $k-1, G-v-w$ ) is true. Finally, if we did not return yet, we return false.

By induction it is a routine to prove that the total time required for this procedure is at most  $O(n^{2k})$ .

**Proposition 31**  $\maxdc(G) = 2$ , if and only if  $G$  is a graph that consists of an even clique and some isolated vertices.

PROOF: Let  $P_1, P_2$  be a left-aligned optimal partition. If  $P_2$  has two vertices, then one of them has an even number of incoming edges from  $P_1$  (since  $d(P_1, P_2)$  is odd). We can put that vertex into  $P_1$  to get a partition  $P_1 = V - \{v\}$  and  $P_2 = \{v\}$ . We denote by  $N(v)$  the neighbours of  $v$ . The following are easy to verify using the fact that  $G[P_1]$  is Eulerian. Suppose  $u_1$  and  $u_2$  are disconnected vertices in  $N(v)$ . The partition  $(P_1 - \{u_1, u_2\}, \{v\}, \{u_1\}, \{u_2\})$  is a good 4-partition of  $G$ , a contradiction. If  $u_1 \in N(v)$  and  $u_2 \in P_1 - N(v)$  are connected, then  $(P_1 - \{u_1, u_2\}, \{v, u_2\}, \{u_1\})$  is a good 3-partition of  $G$ , a contradiction. If  $u_1, u_2 \in P_1 - N(v)$  are connected, then the partition  $(P_1 - \{u_1, u_2\}, \{u_1\}, \{v, u_2\})$  is a good 3-partition of  $G$ , a contradiction.  $\square$

We call a partition *maximal* if all its classes are Eulerian.

**Greedy algorithm** for constructing maximal partitions:

Using the idea from the **Recursive k-dual-critical** method we can construct a maximal left-aligned partition of any graph  $G$ . If there is an odd degree vertex  $v$  or disconnected vertex pair  $v, w$  with  $d(v) + d(w) \equiv 1$  whose deletion *leaves a non-Eulerian graph*, then let this vertex or vertex pair be the last class of the partition, and delete this class from  $G$ . We repeat this step until no such class can be formed. At this point the graph is still non-Eulerian, but the deletion of any odd degree vertex leaves an Eulerian graph.

By the proof of Proposition 31 it is easy to see that the remaining graph consists of an even clique and some isolated vertices. We choose an odd degree vertex  $v$ , and add a new partition class  $\{v\}$ . The rest of the points will provide the first class of the partition. The partition is maximal, since every class spans an Eulerian graph (we made sure to put disconnected vertex pairs in the 2-vertex classes).

By continuously updating vertex degrees, this algorithm runs in  $O(kn^2)$  time.

Let  $\mathcal{P}$  be a fixed maximal left-aligned partition with  $\ell$  classes. We show further possible ways of increasing the number of classes in a maximal partition. Let  $\sigma_0 : P_1 \rightarrow \{2, 3, \dots, \ell, \infty\}$  denote the smallest index  $i$  for which  $d(v, P_i) \equiv 0$  for  $v \in P_1$ . If there is no such index, then we define  $\sigma_0(v) = \infty$ . We define  $\sigma_1 : P_1 \rightarrow \{2, 3, \dots, \ell, \infty\}$  similarly as the smallest index  $i$  for which  $d(v, P_i) \equiv 1$ .

In the following four cases we will be able to increase the number of classes.

- (1) Suppose that  $u$  and  $v$  are distinct isolated vertices in  $G[P_1]$  and  $\sigma_1(u) = \sigma_1(v) = s < \infty$ .
- (2) Suppose that  $u$  and  $v$  are distinct isolated vertices in  $G[P_1]$  and  $\sigma_1(u) = \sigma_1(v) = \infty$  and there is a class  $P_s$  such that  $N(u) \cap P_s \neq N(v) \cap P_s$ .
- (3) Suppose that  $u$  and  $v$  are distinct vertices in the clique of  $G[P_1]$  and  $\sigma_0(u) = \sigma_0(v) = s < \infty$ .
- (4) Suppose that  $u$  and  $v$  are distinct vertices in the clique of  $G[P_1]$  and  $\sigma_0(u) = \sigma_0(v) = \infty$  and  $N(u) \cap P_s \neq N(v) \cap P_s$  for some  $s$ .

In these cases the partition  $(P_1 - \{u, v\}, P_2, \dots, P_s \cup \{u, v\}, P_{s+1}, \dots, P_\ell)$  is a good partition, but the class  $P_s \cup \{u, v\}$  induces a non-Eulerian subgraph. By repeatedly applying Proposition 29 we can split this class into classes of size at most 2. The new partition is a left-aligned maximal partition on more than  $\ell$  classes. This algorithm makes  $O(\ell n^2)$  steps as we can process  $P_i$  for  $i = 2, \dots, \ell$  (to decide whether there are two vertices in  $P_1$  such that one of the rules can be applied with  $s = i$ ) in time  $O(n^2)$ . If one of the rules can be applied, then we can construct a maximal left-aligned partition on at least  $\ell + 1$  classes in the same time window.

**Definition 32 (Equivalent vertices)** A connected vertex pair  $v, w$  is called *connected-equivalent*, or in short *c-equivalent* if  $N(v) - \{w\} = N(w) - \{v\}$ . If  $v$  and  $w$  are disconnected and  $N(v) = N(w)$ , then they will be called *disconnected-equivalent* or *d-equivalent*.

It is easy to verify that both c-equivalence and d-equivalence are equivalence relations.

**Lemma 33** Let  $G$  be a graph and suppose  $k \leq \max_{dc}(G)$ . Let  $\mathcal{P}$  be a maximal left-aligned partition of  $G$  obtained by the **Greedy algorithm** with  $\ell \leq k$  classes that cannot be improved by any of the above operations. If the first class of this partition has size at least  $4k + 1$ , then it contains  $k + 2$  c-equivalent or  $k + 2$  d-equivalent vertices.

**PROOF:** There are at least  $2k + 1$  isolated vertices or a clique of size at least  $2k + 1$  in the graph spanned by the first class of the partition.

Suppose there are  $2k + 1$  isolated vertices. Since operation (1) is not possible, there are at most  $\ell - 1$  isolated vertices with  $\sigma_1(v) < \infty$ , so we have a set  $W$  of at least  $2k + 1 - (\ell - 1) \geq k + 2$  isolated vertices with  $\sigma_1(w) = \infty$ . The partition cannot be improved using operation (2), thus  $N(v) \cap P_i$  is the same set for each  $w \in W$  ( $i = 2, 3, \dots, \ell$ ). Consequently the vertices of  $W$  are d-equivalent.

If we have a clique of size  $2k + 1$  instead, we find a set  $W' \subseteq P_1$ ,  $|W'| = k + 2$  of c-equivalent vertices similarly.  $\square$



**Remark 34** Take a maximal left-aligned partition that cannot be improved using the above operations. For any pair of  $c$ -equivalent (or  $d$ -equivalent) vertices  $u, v \in P_1$  we get that  $\sigma_0(u) = \sigma_0(v) = \infty$  (or  $\sigma_1(u) = \sigma_1(v) = \infty$ ).

**Lemma 35** Let  $G$  be a graph and let  $k \leq \maxdc(G)$ . Let  $\mathcal{P}$  be a maximal left-aligned partition of  $G$  with  $\ell \leq k$  classes that cannot be improved by any of the above operations. If there is a set  $W$  of  $t \geq k + 2$   $c$ -equivalent or  $d$ -equivalent vertices in the first class  $P_1 \in \mathcal{P}$ , then for any  $W' \subseteq W$ ,  $0 \equiv |W'| \leq t - k$ , we have  $\maxdc(G - W') = \maxdc(G)$ .

PROOF: Let  $(R_1, R_2, \dots, R_{\maxdc(G)})$  be a left-aligned optimal partition of  $G$ . It is enough to prove the statement for  $t = k + 2$ , for larger values of  $t$  this argument can be repeated). Observe that (for  $i > 1$ )  $R_i$  cannot contain a pair of  $d$ -equivalent or  $c$ -equivalent vertices, since  $d(\cup_{j=1}^{i-1} R_j, R_i)$  is odd. Thus  $|W \cap R_1| \geq t - k + 1 \geq 3$ , and as we can swap equivalent vertices, we may suppose that  $w_1, w_2 \in W \cap R_1$ . By deleting  $w_1$  and  $w_2$  we get a graph  $G'$  for which the partition  $(R_1 - \{w_1, w_2\}, R_2, \dots, R_{\maxdc(G)})$  is good, thus  $\maxdc(G') \geq \maxdc(G)$ . (We needed 3 equivalent points in the first class so that the class remains non-empty after the deletion.)

We remained to show that  $\maxdc(G') \leq \maxdc(G)$ . For that take a left-aligned optimal partition of  $G'$ . If we put back the vertices  $w_1, w_2$  into the first class, we get a good partition of  $G$ , since the parity of vertex degrees is unchanged.  $\square$

Note that after the deletion of such a vertex set  $W'$  from the first class of  $\mathcal{P}$  we get a good  $\ell$ -partition of  $G - W'$ .

**Theorem 36** Given  $G(V, E)$  and  $k$  as input, there is an algorithm that either finds a good  $k$ -partition or it yields a subset  $K \subseteq V$  with the following properties:

- (1)  $|K| \leq 6k$
- (2)  $\maxdc(G[K]) \geq k$  if and only if  $\maxdc(G) \geq k$
- (3) For any good  $k$ -partition of  $G[K]$  the partition obtained by adding the vertices of  $V - K$  to the first class we get a good  $k$ -partition of  $G$ .

The algorithm runs in  $O(k^2 n^2)$  time.

PROOF: We make an  $\ell_0$ -part maximal left-aligned partition using the **Greedy algorithm**. After that we can improve the partition using the improvement operations on page 8, resulting in a partition of  $\ell \geq \ell_0$  classes. If  $\ell \geq k$ , then we can obtain a good  $k$  partition (by contracting the first few classes if needed). The algorithm writes this partition to the output. If  $\ell < k$ , and  $|V| > 6k$ , then the first partition has at least  $6k - 2(\ell - 1) > 4k + 1$  vertices. By Lemma 33 we have a  $c$ - or  $d$ -equivalence class of size at least  $k + 2$ . Lemma 35 allows us to delete vertices from the first class, and thereby decrease its size under  $4k + 1$ . The new vertex set  $K \subsetneq V$  that we obtained has at most  $6k$  vertices, and  $\maxdc(G[K]) = \maxdc(G)$ .

Now we estimate the time needed for the computations. Finding a maximal partition using the **Greedy algorithm** can be done in  $O(kn^2)$  time. After this we make the possible improvements from page 8. All improvements can be done in  $O(k^2 n^2)$  time (since we can do at most  $k$  improvements, and each of them takes  $O(kn^2)$  time). In the end we delete a set of vertices, which takes no more than  $O(n^2)$  time.  $\square$

The graph spanned by  $K$  is called the *kernel* of the problem. For the kernel we can run Algorithm **Recursive k-dual-critical**.

**Corollary 37** There is a fixed parameter tractable (FPT) algorithm, that given  $G$  and  $k$ , computes a good  $k$ -partition, or concludes that none exists. The algorithm runs in  $O(k^2 n^2 + (6k)^{2k})$  time.

**Corollary 38** Using the above algorithm repeatedly for  $k = 1, 2, \dots, \maxdc(G) + 1$  it is possible to compute  $\maxdc(G)$  in  $O(n^4 + n^2(6 \maxdc(G))^{2 \maxdc(G)})$  time.

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